Derivatives, Images, and Gradients

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Marist School

Definition of a Derivative

- Rate of Change over between 2 or move variables.
- Expressed as a ratio

$$\frac{\Delta y}{\Delta x}$$

- Slope of a Line tangential to a point on a continuous function
- Foundation of Calculus / Physics / AI
- Continuous and Discrete applications.

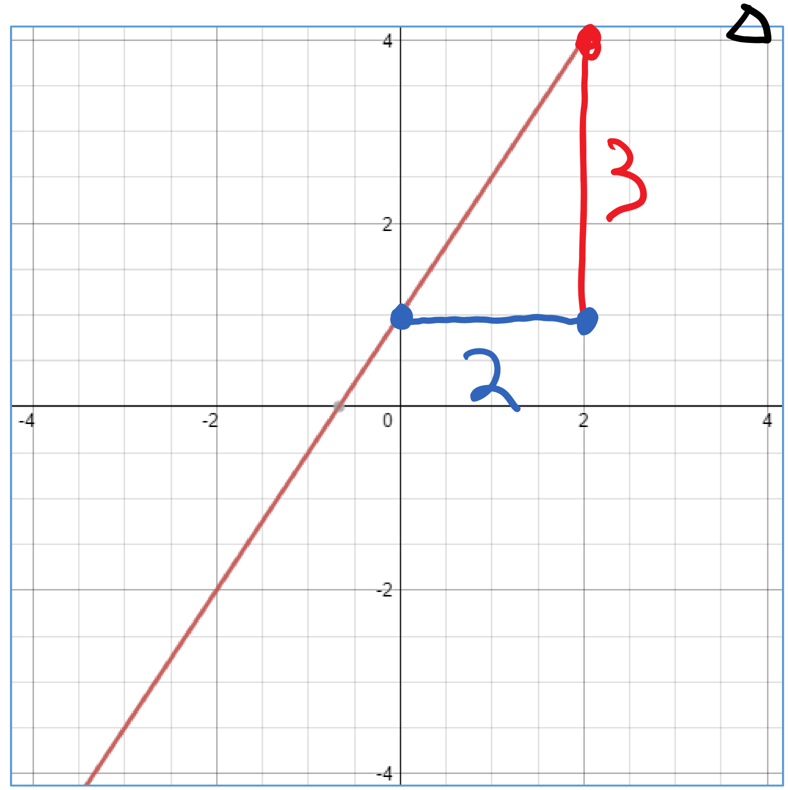
You have already used Derivatives with Linear Functions.

- The Derivative in a linear function is the same as the slope. $\frac{\Delta y}{\Delta x}$

$$f(x) = \frac{3}{2}x + 1$$

$y = mx + b$

$\text{slope} = m = \frac{3}{2}$



Mathematical Definition of a Derivative (Continuous Functions)

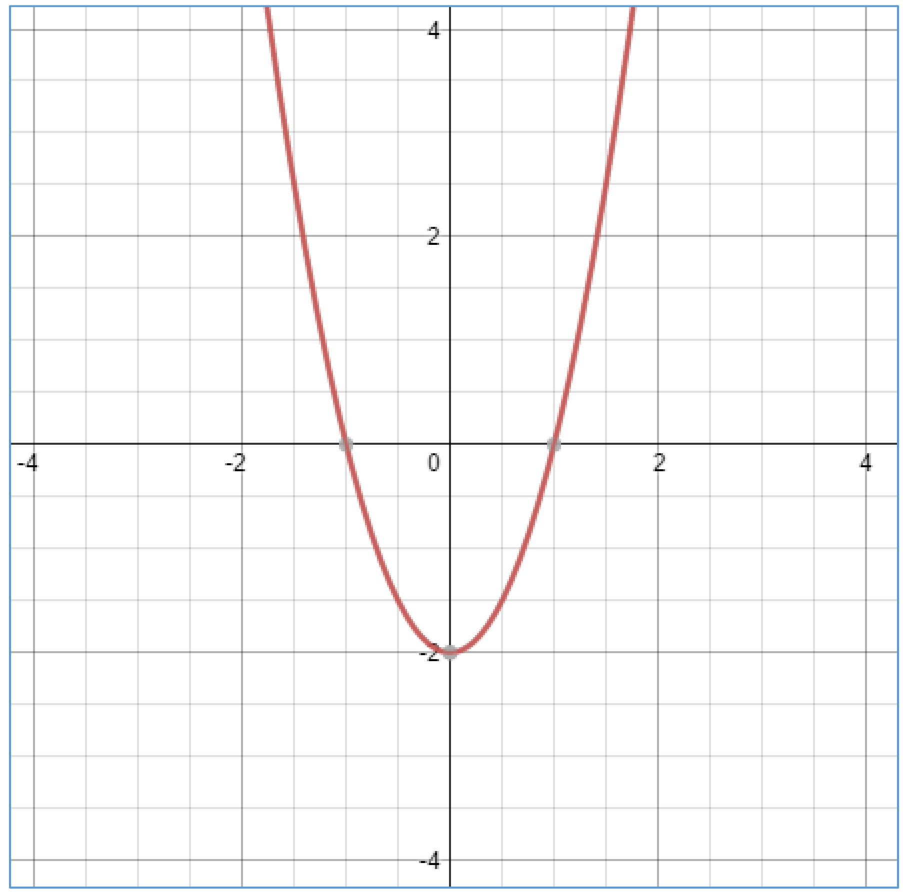
- The derivative of a *f(x)* with respect to x is the function *f'(x)* and is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For most continuous functions, we can compute the derivatives with formulas.

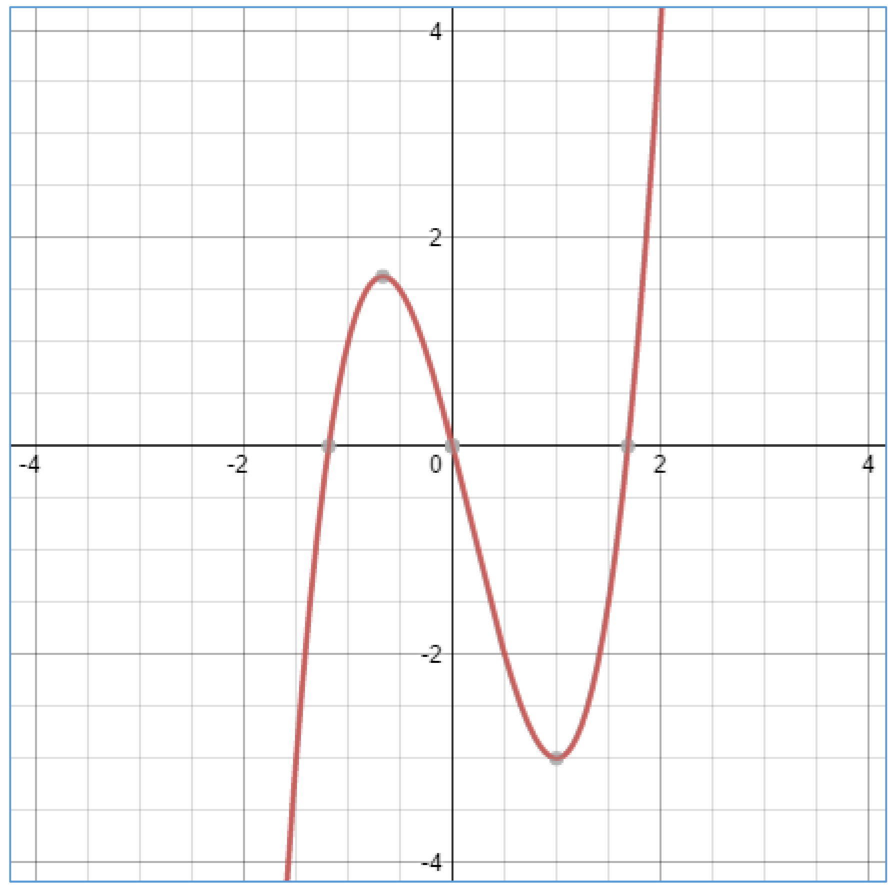
$$f(x) = 2x^2 - 2$$

$$\frac{dy}{dx} = ?$$



Another Continuous Function

$$f(x) = 2x^3 - x^2 - 4x$$

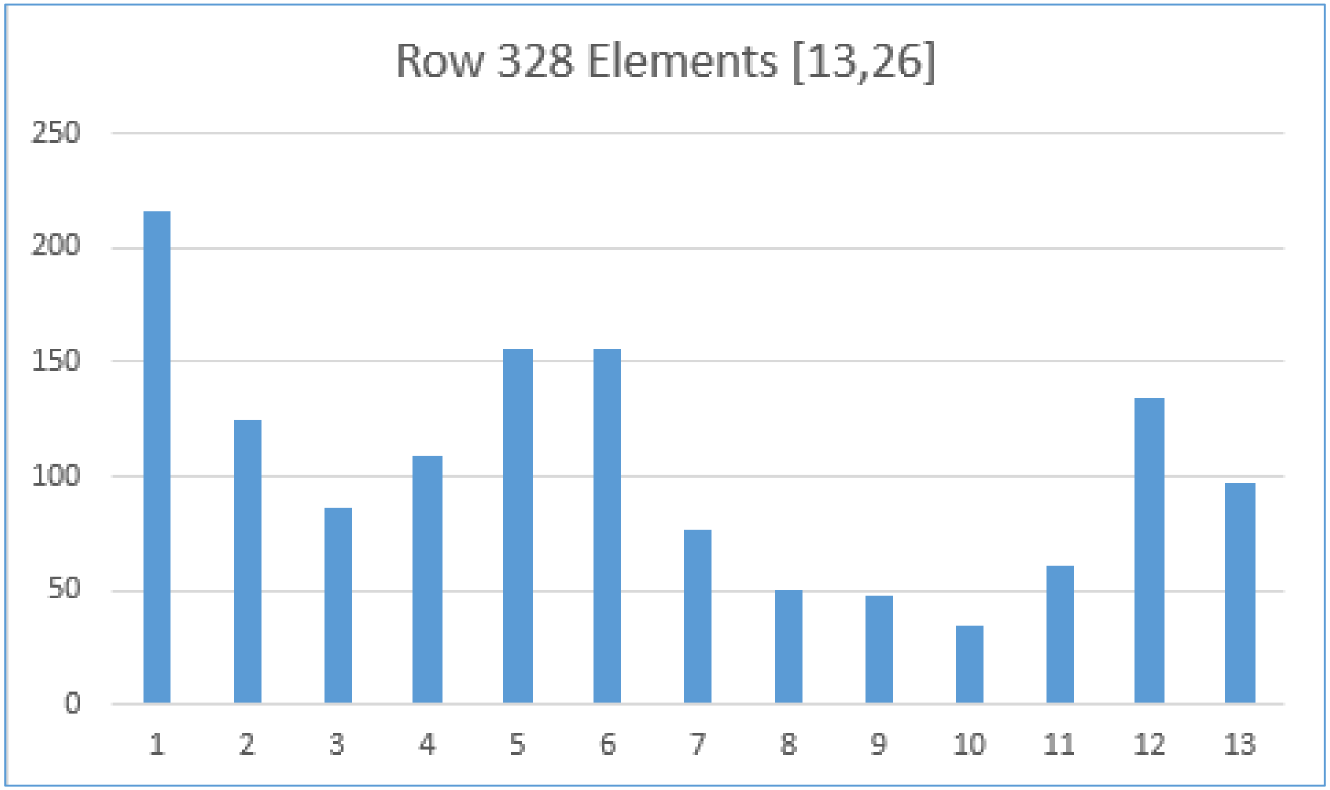


What about discrete data?

- In computing – (such as images or sound), Data is discrete.
  - Comes in Chunks
  - May not be continuous or defined by formula
- We can still compute and use derivatives.
  - How? Element wise subtraction!

Example Data: Row 328

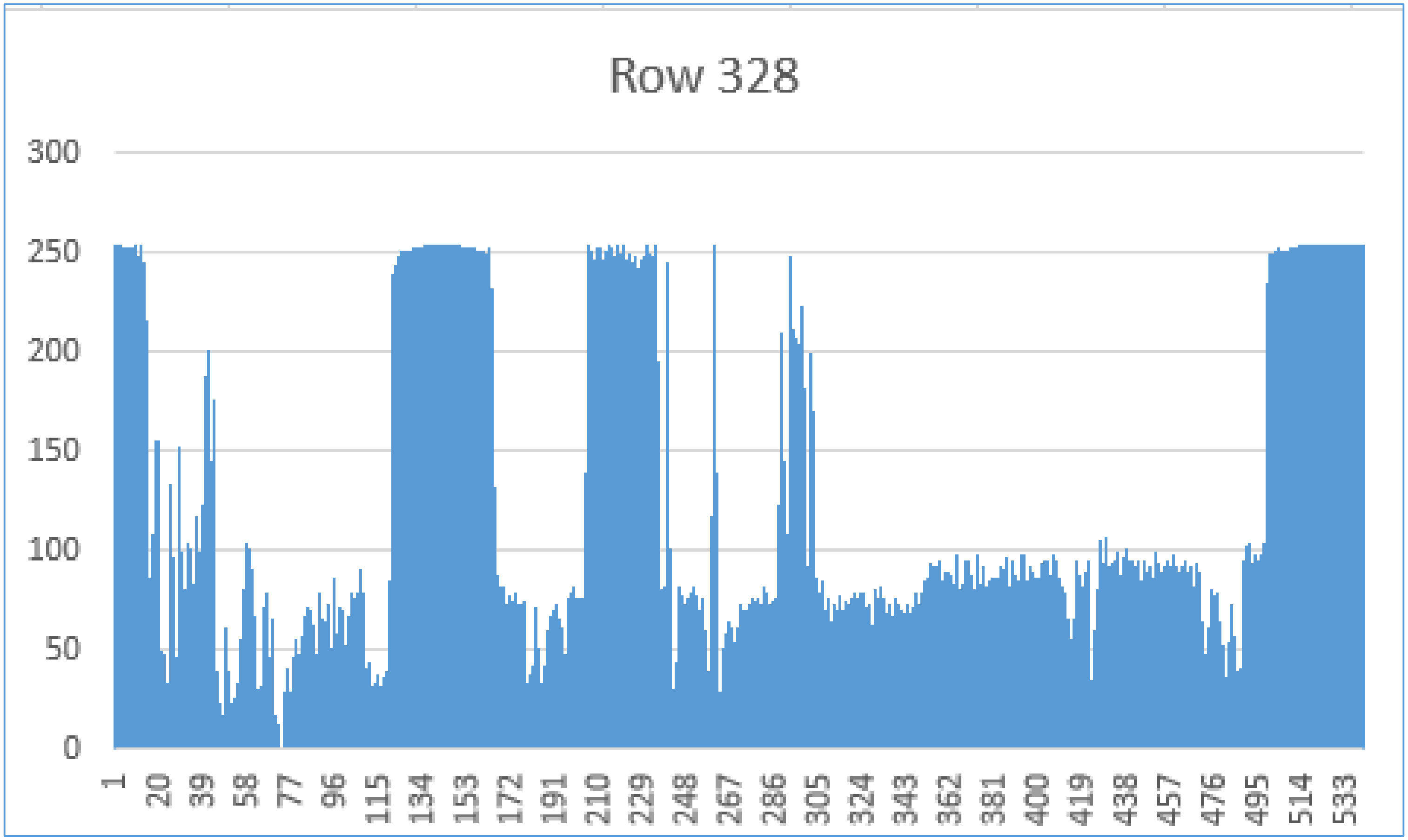
What does the rate of change tell us about this data?



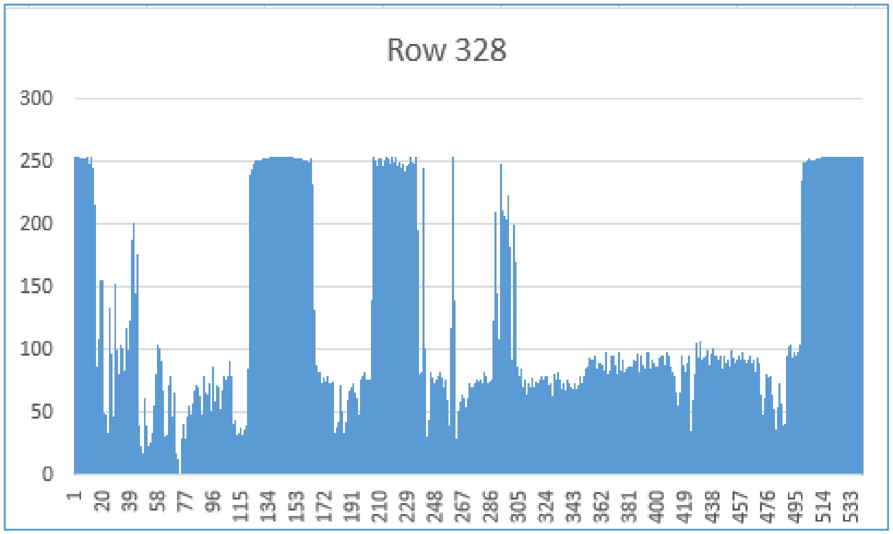
[216, 125, 86, 109, 156, 156, 76, 50, 48, 34, 61, 134, 97]

Example: Row 328 [:]

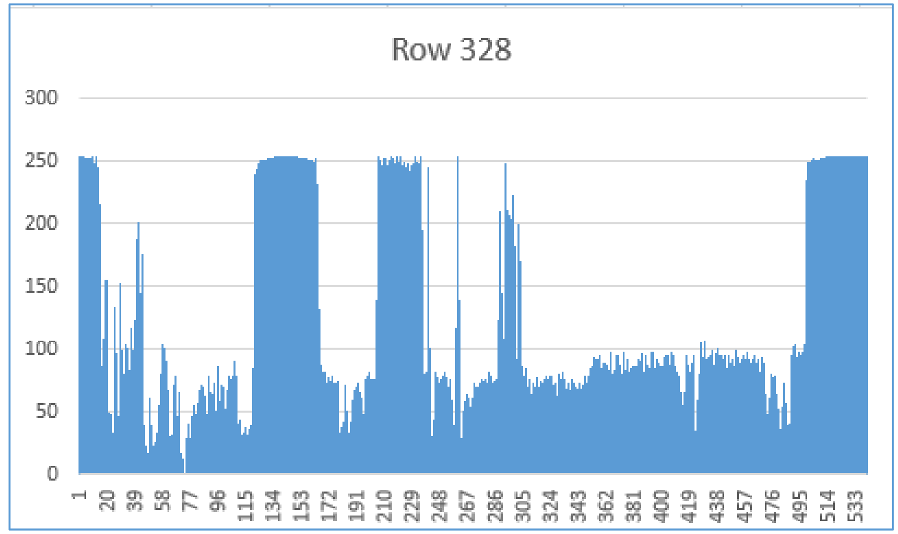
What could the derivative of this data tell us? What information can we gather?

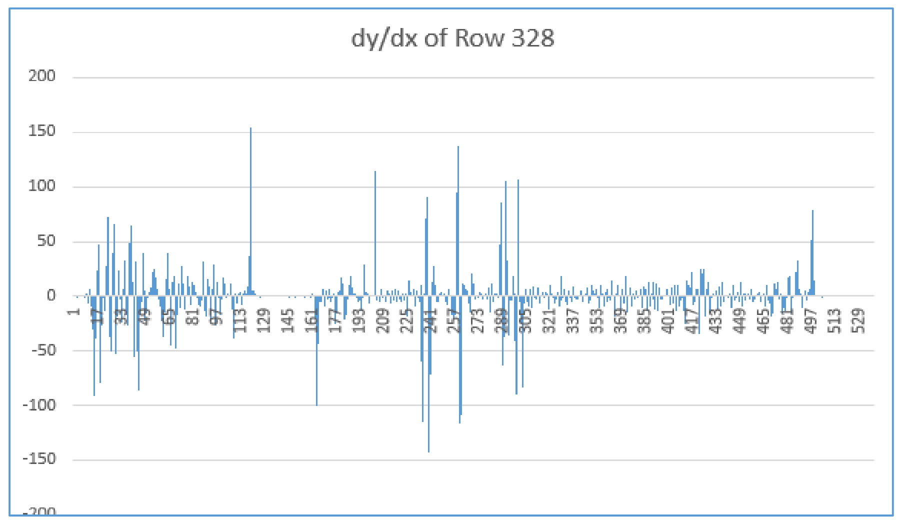


What can we get from the derivative?

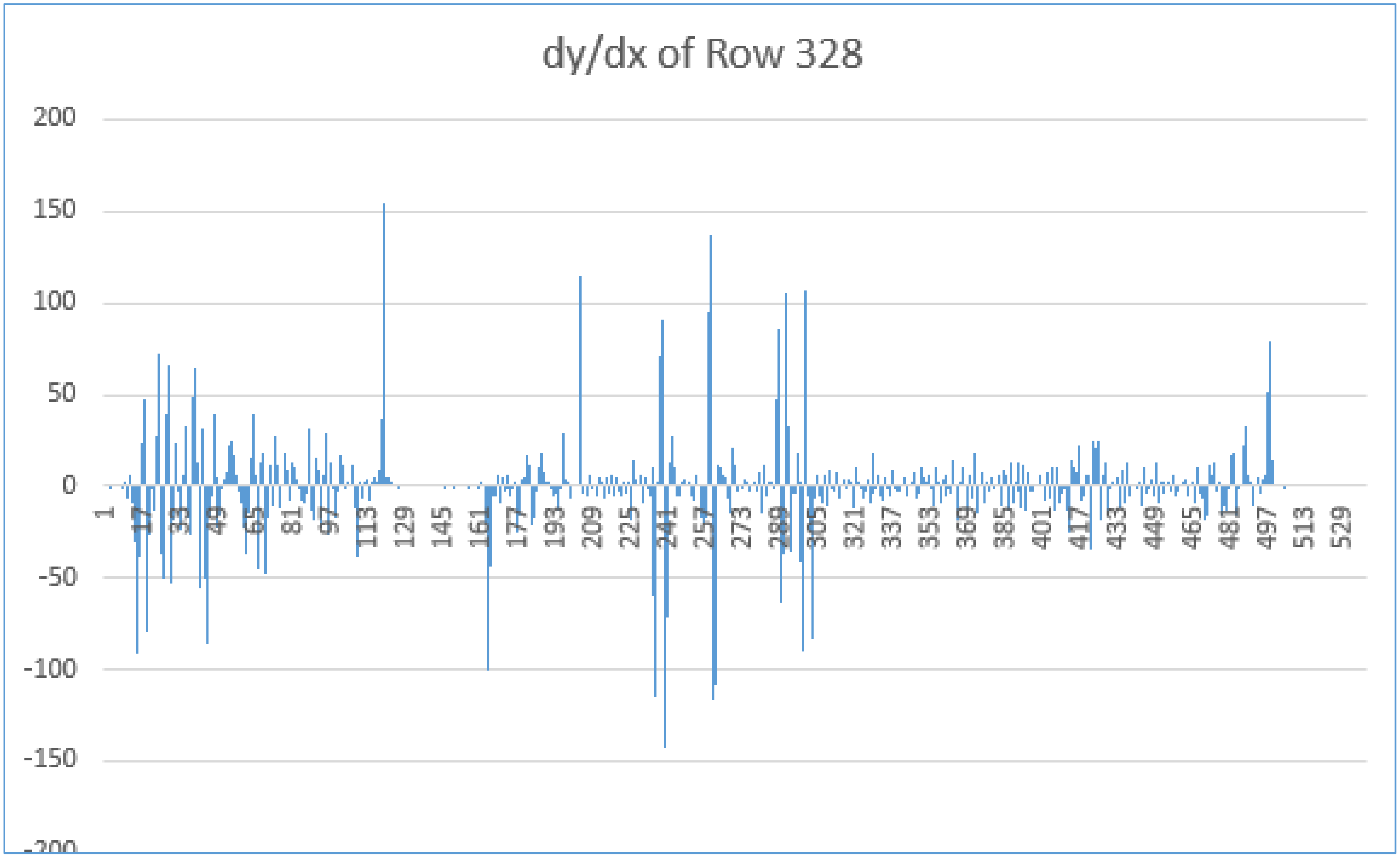


What can we get from the derivative?





Closer Look . . .



dy/dx of Row 328
200
150
100
50
0
-50
-100
-150
-200
1
17
33
49
65
81
97
113
129
145
161
177
193
209
225
241
257
273
289
305
321
337
353
369
385
401
417
433
449
465
481
497
513
529

Derivatives for Two Inputs: Image Functions

- Recall the Intensity Function:

$$I(x, y)$$

- Returns the intensity of a pixel value at (x,y)

- We can take two derivatives
  - For change in intensity / change in x
  - For change in intensity / change in y

Continuous Function *I(x, y)*

- Two Derivatives: Formal Definition for x direction:

$$\frac{\Delta I(x,y)}{\Delta x} = \lim_{h \to 0} \frac{f(x+h,\ y) - f(x,y)}{h}$$

- Formal Definition for y direction:

$$\frac{\Delta I(x,y)}{\Delta y} = \lim_{h \to 0} \frac{f(x,\ y+h) - f(x,y)}{h}$$

But . . Images are discrete. Thus, we modify for discrete data:

- X Direction

$$\frac{\Delta I(x,y)}{\Delta x} = \frac{f(x+1,\ y) - f(x,y)}{1}$$

$$\frac{\Delta I(x,y)}{\Delta x} = f(x+1,\ y) - f(x,y)$$

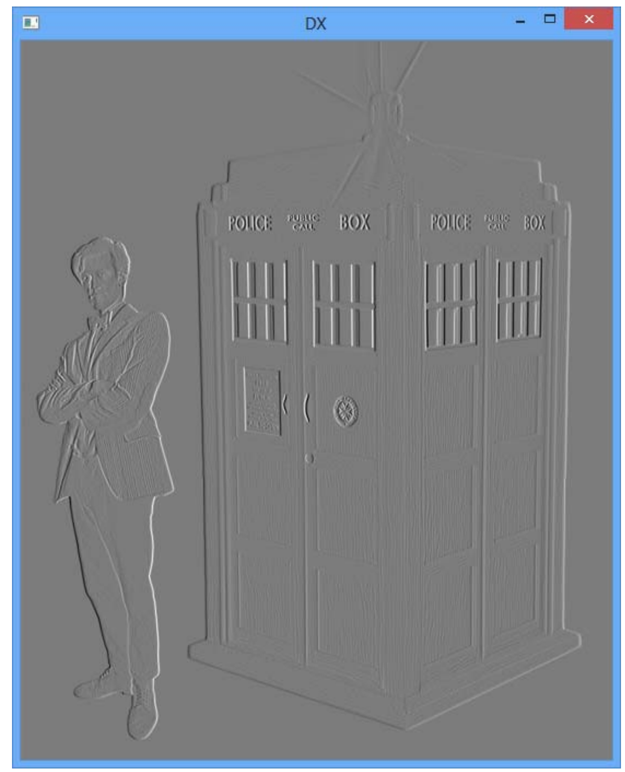
But . . Images are discrete. Thus, we modify for discrete data:

- Y Direction

$$\frac{\Delta I(x, y)}{\Delta y} = \frac{f(x+1,\ y) - f(x, y)}{1}$$

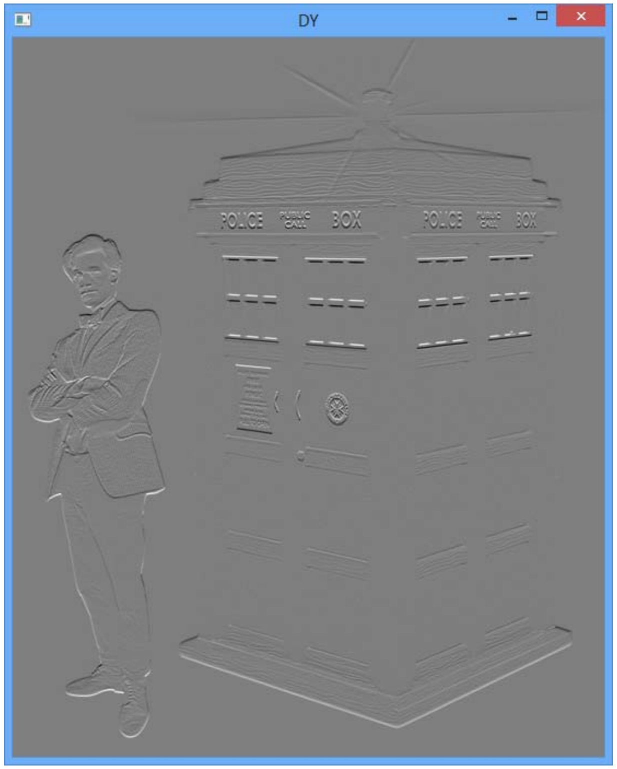
$$\frac{\Delta I(x,y)}{\Delta y} = f(x,\ y+1) - f(x, y)$$

What does this look like in an actual image?



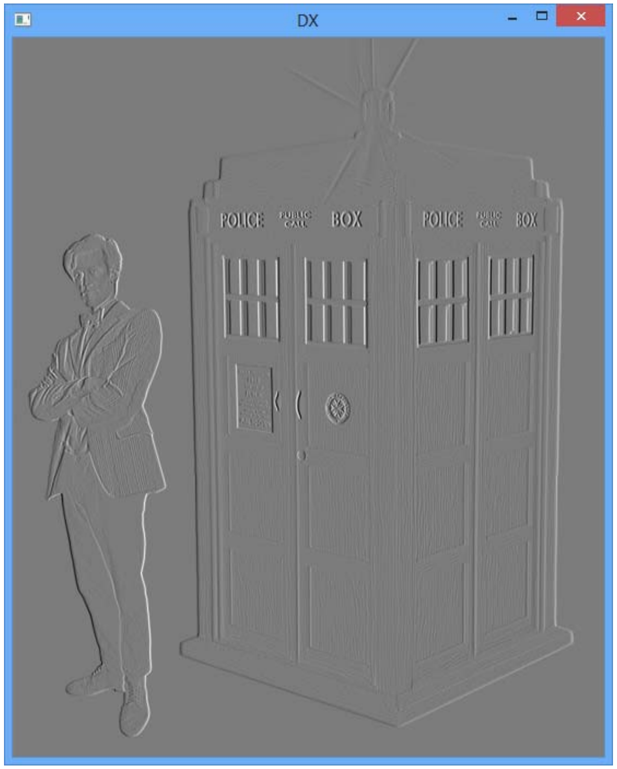
$$\frac{\Delta I(x,y)}{\Delta x} = f(x+1,\ y) - f(x,y)$$

What does this look like in an actual image?

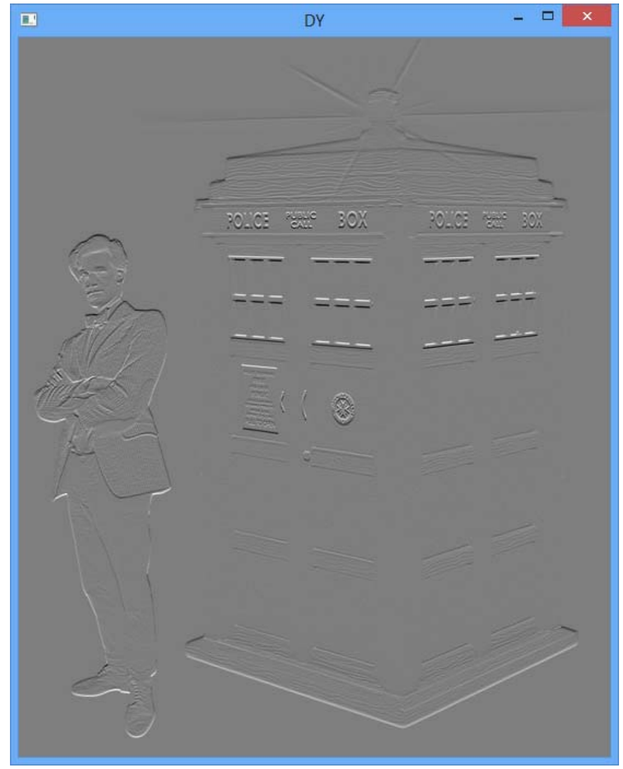


$$\frac{\Delta I(x,y)}{\Delta y} = f(x,\ y+1) - f(x,y)$$

Side by Side comparison . . .



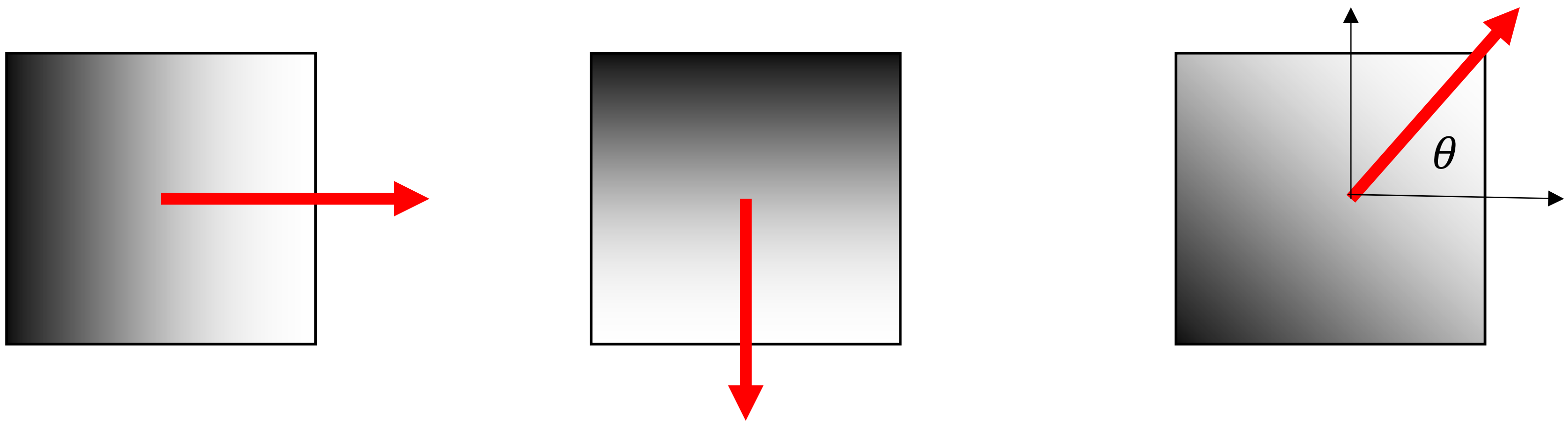
$$\frac{\Delta I(x,y)}{\Delta x}$$



$$\frac{\Delta I(x,y)}{\Delta y}$$

Application: Image Gradients

$$\nabla I = \left[\frac{\Delta I}{\Delta x}, \frac{\Delta I}{\Delta y}\right]$$



$$\nabla I = \left[\frac{\Delta I}{\Delta x}, 0\right] \qquad \nabla I = \left[0, \frac{\Delta I}{\Delta y}\right] \qquad \nabla I = \left[\frac{\Delta I}{\Delta x}, \frac{\Delta I}{\Delta y}\right]$$

Gradient points in direction of the most rapid change in Intensity.

Mathematical Definitions:

The gradient of an Image: $\nabla I = \left[\frac{\Delta I}{\Delta x}, \frac{\Delta I}{\Delta y}\right]$

The gradient Direction: $\theta = \tan^{-1}\left(\frac{\Delta I}{\Delta y} / \frac{\Delta I}{\Delta x}\right)$

The gradient Magnitude (Edge Strength): $||\nabla I|| = \sqrt{\left(\frac{\Delta I}{\Delta x}\right)^2 + \left(\frac{\Delta I}{\Delta y}\right)^2}$

Derivate in x with discrete data (One Dimension):

$V =$

125	86	109	156	156	76	50	48

$\Delta V =$

Derivate in x with discrete data (One Dimension):

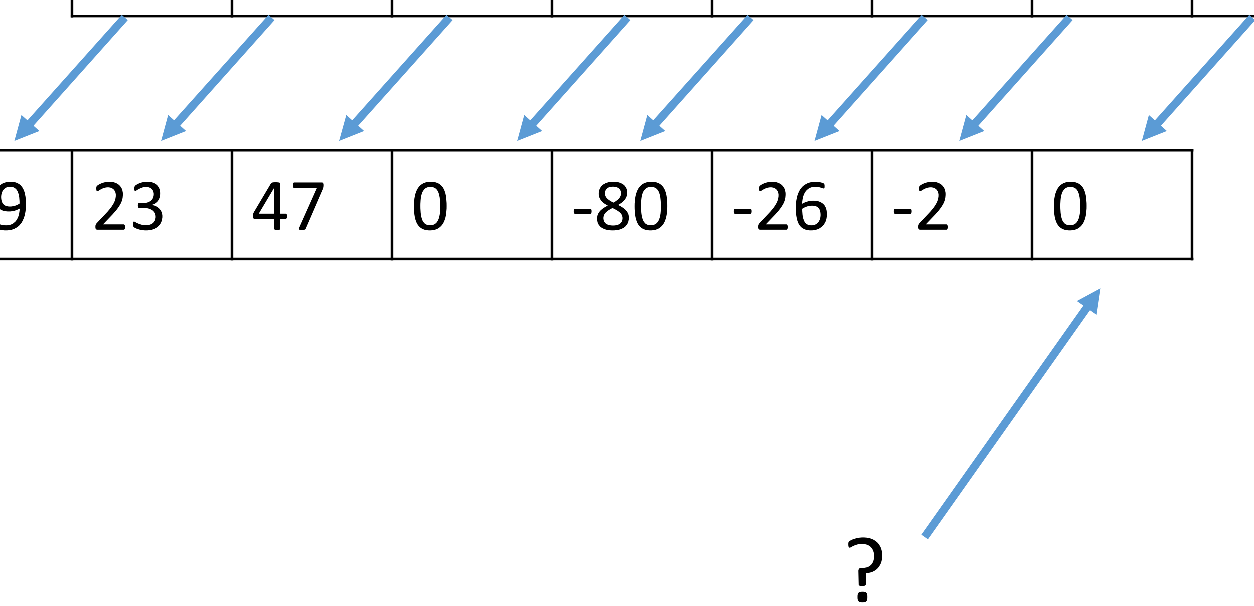
$V =$

125	86	109	156	156	76	50	48

125	86	109	156	156	76	50	48

$\Delta V =$

-39	23	47	0	-80	-26	-2	0



Python Example: Derivative on one dimension

*derivative_practice.py - C:/Users/michaudc/OneDrive/BCT 410 Computational Perception/Uni...

File Edit Format Run Options Windows Help

```python
# Derivative in One Dimension

import numpy as np

# Define Array
V = np.array([125, 86, 109, 156, 156, 76, 50, 48])

# Change Datatype to float
V = V.astype(np.float32)

# Make Blank Array same size as V
dV = np.zeros(len(V), np.float32)

# Compute Derivative
dV[0:len(dV)-1] = V[1:len(V)] - V[0:len(V)-1]

# Output
print "V: ", V
print "dV: ", dV
```

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